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New Lax pair for 2D dimensionally reduced gravity

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Abstract

Using a Lax pair based on the affine $SL(2,\mathbb{R})$ Kac–Moody algebra, we solve two-dimensional reduced vacuum Einstein's equations. We obtain explicit determinant formulae for metric coefficients with no quadrature left. This Lax connection also allows a new approach to the Poisson algebra of two-dimensional reduced gravity. In particular, we show that it leads to pure c-number r-matrices and modified Yang–Baxter equations. We explain how one can construct classical observables within this framework.

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1. Introduction

The 2-surface commuting Killing vector reduction of source-free general relativity has the particular property of exhibiting an infinite-dimensional symmetry group known as the Geroch group [1], later identified by Julia [2] as the affine $SL(2,\mathbb{R})$ Kac–Moody group. The reduced Einstein's equations (called Ernst's equations [6]) are known to be integrable since the works of Belinskii, Zakharov and Maison [3, 4]. Because we can apply techniques developed for integrable systems this model, equivalent to $SL(2,\mathbb{R})/SO(2)$ coset space σ-models coupled to two-dimensional gravity and a dilaton, has been often used as a toy model for the quantization of gravity. An extensive study has been made in this domain by Julia, Korotkin, Nicolai and Samtleben [7, 8, 10].

Nevertheless, all of these previous works are based on a moving-pole approach which has the disadvantage of blurring the relation between the symmetry and integrability of the model. Moreover, previous metric formulae are not purely algebraic (even those of Letelier [9]) and there is at least one integral which can be and has been, in a few instances [3], evaluated case by case. Concerning the Poisson algebra, no Yang–Baxter equations were found.

Another Lax connection based on the affine $SL(2,\mathbb{R})$ Kac–Moody algebra was proposed in [11] that solves these problems. The aim of this paper is to give a brief survey of what has been developed from this new Lax connection [11–13]. In particular, we will show how using dressing transformations, we can obtain determinant formulae giving expressions of metric

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elements with no quadrature left. We shall also see that the Poisson brackets of the connection lead to pure c-number-modified Yang–Baxter equations.

This paper is organized as follow. Section 2 sums up some basic results concerning two-dimensional (2D) reduced gravity such as Ernst's equations and Kramer–Neugebauer duality. Section 3.2 introduces the Lax connection that will be used in the rest of our paper:

$$\begin{aligned} A_x &= \tfrac{1}{2}\rho^{-1}(\partial_t\rho + \partial_x\rho)E_+ - \tfrac{1}{2}\rho^{-1}(\partial_x\rho - \partial_t\rho)E_- + \tfrac{1}{2}\left(P_{xa} + P_{ta}\right)T^a\lambda \\ &\quad + \tfrac{1}{2}\left(P_{xa} - P_{ta}\right)T^a\lambda^{-1} + Q_{x\alpha}T^\alpha - \tfrac{1}{2}\partial_t\widehat{\sigma}k \\ A_t &= \tfrac{1}{2}\rho^{-1}(\partial_t\rho + \partial_x\rho)E_+ + \tfrac{1}{2}\rho^{-1}(\partial_x\rho - \partial_t\rho)E_- + \tfrac{1}{2}\left(P_{xa} + P_{ta}\right)T^a\lambda \\ &\quad - \tfrac{1}{2}\left(P_{xa} - P_{ta}\right)T^a\lambda^{-1} + Q_{t\alpha}T^\alpha - \tfrac{1}{2}\partial_x\widehat{\sigma}k. \end{aligned}$$

Section 4 is devoted to the dressing group, and how it can lead to simple formulae for metric elements. We also expose vertex operators used as highest-weight representations for explicitly evaluate expressions. In section 5, we explain how duality can solve the factorization problem and we will give the determinant formulae for the metric and its dual. Finally, in section 6, we will review the Poisson algebra of the model, the quite surprising c-number r-matrices and associated modified Yang–Baxter equations. We will also show how to construct classical observables and difficulties which occur when we try to evaluate their Poisson brackets.

2. Properties of 2D reduced gravity

In this section, we will review a few basic facts concerning 2D reduced gravity which corresponds to the case where there are two surface orthogonal commutating Killing vectors. We assume that these two Killing vectors are spacelike. Analytic continuation provides a way to translate the corresponding results into the case where there is one spacelike Killing vector and one timelike Killing vector. Under these assumptions, the metric can be brought to the form

$$\mathrm{d}s^2 = \rho^{1/2}\mathrm{e}^{2\widehat{\sigma}}(-\mathrm{d}t^2 + \mathrm{d}z^2) + \rho S_{ij}(z,t)\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{1}$$

where $\rho(z,t)$ is called the dilaton and $\widehat{\sigma}(z,t)$ is the conformal factor. The symmetric 2×2 matrix S, normalized by $\det(S) = 1$, can be written as $S = \mathcal{V}^t\mathcal{V}$, where $\mathcal{V}$ is an element of $SL(2,\mathbb{R})$. $\mathcal{V}$ is equivalent to the internal zweibein, up to a $\sqrt{\rho}$ factor. Note that the metric is invariant when acting on $\mathcal{V}$ on the right with a global $SL(2,R)$ reparametrization and on the left with a local $SO(2)$ transformation.

To write the corresponding vacuum Einstein equation (the so-called Ernst equations) in a gauge covariant way, we can formulate them as a nonlinear sigma model on $SO(2)\backslash SL(2,R)$. We will use the decomposition of $sl(2,R) = h \oplus g$, where $h = so(2)$ is the maximal compact subalgebra of $sl(2,R)$. We denote each component of this connection as $P_x + Q_x = \mathcal{V}\partial_x\mathcal{V}^{-1}$ and $P_t + Q_t = \mathcal{V}\partial_t\mathcal{V}^{-1}$, where P is an element of g and Q belongs to h. With these objects, Einstein's equation can be written as

$$\partial_x Q_t - \partial_t Q_x + [Q_x, Q_t] + [P_x, P_t] = 0 \tag{2}$$

$$\partial_x P_t + [Q_x, P_t] = \partial_t P_x + [Q_t, P_x] \tag{3}$$

$$\partial_x(\rho P_x) + [Q_x, \rho P_x] = \partial_t(\rho P_t) + [Q_t, \rho P_t] \tag{4}$$

$$\left(\partial_t^2 - \partial_x^2\right)\rho = 0 \tag{5}$$

$$((\partial_t \pm \partial_x)\rho)(\partial_t \pm \partial_x)\widehat{\sigma} = -\rho\tfrac{1}{2}\,\mathrm{tr}\left((P_x \pm P_t)^2\right). \tag{6}$$

A very useful gauge choice corresponds to the case where $\mathcal{V}$ is triangular

$$\mathcal{V} = \begin{pmatrix} \Delta^{-1/2} & 0 \\ -N\Delta^{1/2} & \Delta^{1/2} \end{pmatrix}.$$

The corresponding metric is

$$\mathrm{d}s^2 = \rho^{-1/2}\mathrm{e}^{2\widehat{\sigma}}\,(\mathrm{d}z^2 - \mathrm{d}t^2) + \rho\Delta^{-1}\,\mathrm{d}x^2 + \rho\Delta(\mathrm{d}y - N\,\mathrm{d}x)^2. \tag{7}$$

Thus, the metric is defined by the fields $(\rho, \widehat{\sigma}, \Delta, N)$, up to some transformations (translations on $\widehat{\sigma}$ and N, dilatation on Δ).

One of the most remarkable properties of 2D reduced gravity is the existence of a duality relation known as the Kramer–Neugebauer duality. Replacing the original fields $(\rho, \widehat{\sigma}, \Delta, N)$ by their dual $(\rho^*, \widehat{\sigma}^*, \Delta^*, N^*)$ defined by

$$\rho^* = \rho \qquad \Delta^* = 1/(\rho\Delta) \qquad \Delta^*\partial_\nu N^* = \epsilon_{\nu\alpha}\Delta\partial_\alpha N \qquad \Delta^*\mathrm{e}^{4\widehat{\sigma}^*} = \Delta\mathrm{e}^{4\widehat{\sigma}} \tag{8}$$

we obtain a new metric $\mathrm{d}s_*^2$, which is also a solution of the vacuum Einstein's equations. A way to explain why this model is solvable is to verify that the global $SL(2, R)$ transformations do not commute with the duality operation. Thus, starting from a known solution and by applying successive reparametrization and duality operations, we can generate an infinite number of solutions.

3. The Lax connection

3.1. Some algebraic tools

Before dealing with the Lax connection, we shall introduce the algebra we will use. Consider the $sl(2, R)$ affine Kac–Moody algebra defined by the commutation relations

$$\left[X\lambda^n, Y\lambda^m\right] = [X, Y]\lambda^{m+n} + \tfrac{1}{2}nk\,\mathrm{tr}(XY)\delta_{n+m,0}. \tag{9}$$

We twist this algebra with the order-two automorphism that leaves h invariant. It means that for some element $X\lambda^n$, if n is even, then X is an element of h, otherwise X is an element of g. We will denote this algebra as $\mathcal{H}_{taf}$, H_{taf} the associated group and $\mathcal{B}_\pm = \{X\lambda^{\pm n} | n > 0\} \oplus k\mathbb{C}$ the two Borel subalgebras. We will use the following notation for the generators: T^α with a Greek index corresponds to the generators of h and T^a with a Latin index corresponds to the generators of g.

Actually, we will use the semi-direct product of $\mathcal{H}_{taf}$ with the Virasoro algebra. We recall that the commutation relations for the Virasoro algebra are

$$[L_n, L_m] = (n - m)L_{m+n} + n(n^2 - 1)\frac{c}{12}\delta_{n+m,0}$$

and the crossed Lie bracket is $[L_n, X\lambda^m] = -\frac{1}{2}mX\lambda^{n+m}$. For convenience, we introduce a particular notation for two elements of the Virasoro algebra $E_\pm = L_0 - L_{\pm 1}$, which verify the commutation relation $[E_+, E_-] = E_+ + E_-$. We choose the following basis $h = \{\sigma^+ - \sigma^-\}$ and $g = \{\sigma^+ + \sigma^-, \sigma^z\}$ with the convention

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for the Pauli matrices. We shall use highest-weight representations during our calculations. These representations possess two fundamental highest vector $|\Lambda_\pm\rangle$ characterized by

$$\begin{gathered}\left(\sigma^+ - \sigma^-\right)|\Lambda_\pm\rangle = \pm\tfrac{1}{2}\mathrm{i}|\Lambda_\pm\rangle \qquad k|\Lambda_\pm\rangle = |\Lambda_\pm\rangle \qquad \text{and} \qquad L_0|\Lambda_\pm\rangle = h_{\Lambda_\pm}|\Lambda_\pm\rangle \\ X\lambda^n|\Lambda_\pm\rangle = L_n|\Lambda_\pm\rangle = 0 \qquad \text{for} \quad n > 0.\end{gathered} \tag{10}$$

In the same way, we define the dual vectors $\langle\Lambda_\pm|$ such that $\langle\Lambda_\pm|X\lambda^n = \langle\Lambda_\pm|L_n = 0$ for $n < 0$.

3.2. The Lax connection

As we mentioned before, the model is integrable. This means that an auxiliary linear system

$$(\partial_t + A_t)\,\Psi = 0 \qquad \text{and} \qquad (\partial_x + A_x)\,\Psi = 0 \tag{11}$$

can be found such that the zero-curvature condition $[\partial_t + A_t, \partial_x + A_x] = 0$ reproduces the equations of motion. The expression of the Lax connection components that fulfils this requirement, is

$$\begin{aligned} A_x = {} & \tfrac{1}{2}\rho^{-1}(\partial_t\rho + \partial_x\rho)E_+ - \tfrac{1}{2}\rho^{-1}(\partial_x\rho - \partial_t\rho)E_- + \tfrac{1}{2}\,(P_{xa} + P_{ta})\,T^a\lambda \\ & + \tfrac{1}{2}\,(P_{xa} - P_{ta})\,T^a\lambda^{-1} + Q_{x\alpha}T^\alpha - \tfrac{1}{2}\partial_t\widehat{\sigma}k \end{aligned} \tag{12}$$

$$\begin{aligned} A_t = {} & \tfrac{1}{2}\rho^{-1}(\partial_t\rho + \partial_x\rho)E_+ + \tfrac{1}{2}\rho^{-1}(\partial_x\rho - \partial_t\rho)E_- + \tfrac{1}{2}\,(P_{xa} + P_{ta})\,T^a\lambda \\ & - \tfrac{1}{2}\,(P_{xa} - P_{ta})\,T^a\lambda^{-1} + Q_{t\alpha}T^\alpha - \tfrac{1}{2}\partial_x\widehat{\sigma}k. \end{aligned} \tag{13}$$

We shall need a seed solution from which we will generate new ones. The simplest case we can consider is when $P = Q = \widehat{\sigma} = 0$. This solution is called the vacuum solution and one can easily prove it is dual to the Minkowski flat space one. The associated wavefunction Ψ_V with a specific choice of normalization, is given by

$$\begin{aligned} \Psi_V &= \left(\frac{\rho + \int \partial_t\rho + b}{2\rho}\right)^{E+} \left(\frac{\rho + \int \partial_t\rho + b}{b'}\right)^{E-} \\ &= \left(\frac{\rho - \int \partial_t\rho + c}{2\rho}\right)^{-E-} \left(\frac{\rho - \int \partial_t\rho + c}{c'}\right)^{-E+}. \end{aligned} \tag{14}$$

Note that Ψ_V develops only on the Virasoro algebra. Actually, all the coordinate dependence will be included in Ψ_V. The Virasoro algebra is just a way to encode this dependence of the metric. In particular, if we conjugate Ψ with Ψ_V^{-1} (see [12]), we restore this dependence into the poles (moving poles) and obtain a Lax connection similar to those of [5, 8].

4. Dressing group and vertex operators

4.1. Dressing the vacuum

Dressing transformations are non-local gauge transformations that preserve the form of the connection. Let g be a constant element of H_{taf} and its factorization on the two Borel subalgebra $g = g_-^{-1}g_+$ with $g_\pm \in \exp(\mathcal{B}_\pm)$ and consider the transformation

$$\Psi \quad \rightarrow \quad \Theta_- \cdot \Psi \cdot g_-^{-1} = \Theta_+ \cdot \Psi \cdot g_+^{-1}$$

with $\Theta_\pm = \left(\Psi g \Psi^{-1}\right)_\pm$ and $\Theta_-^{-1}\Theta_+ = \Psi g \Psi^{-1}$. This transformation acts on the connection as a non-local gauge transformation

$$A \quad \rightarrow \quad \Theta_\pm A \Theta_\pm^{-1} - \partial\Theta_\pm \cdot \Theta_\pm^{-1}$$

and it can be shown that it preserves the zero-curvature condition and the form of the Lax connection. To generate solutions we just have to dress a given wavefunction with an element of H_{taf}. One can show that the phase space contains only one dressing orbit, thus we just need a first seed metric to obtain all metrics. We draw the reader's attention to the problem of factorization arising when writing $g = g_-^{-1}g_+$. We can write $g_\pm$ as

$$g_\pm = \exp\left(\pm\eta\frac{k}{2}\right)\exp\left(-\frac{\varphi_\pm}{2}\,(\sigma_+ - \sigma_-)\right) \quad (\text{degree} > 1). \tag{15}$$

Only the difference $\varphi = \varphi_+ - \varphi_-$ is fixed, which is just a consequence of the local $SO(2)$ gauge invariance. Note that ρ is unchanged by the dressing transformation and if we dress the vacuum wavefunction, the new conformal factor $\widehat{\sigma}$ is equal to η. It is possible to evaluate P, Q from the previous formulae but we can have much more. The main advantage of this method is to provide a quite direct way to find quantities that define the metric. Consider the element $\widehat{\mathcal{E}} = \widehat{\mathcal{E}}_+ - \widehat{\mathcal{E}}_-$ of $\mathcal{H}_{taf}$ with

$$\widehat{\mathcal{E}}_\pm = \pm\left[(\sigma^+ - \sigma^-)\otimes\left(\frac{1+\lambda^{\pm 2}}{1-\lambda^{\pm 2}}\right) - (\sigma^+ + \sigma^-)\otimes\left(\frac{2\lambda^{\pm 1}}{1-\lambda^{\pm 2}}\right)\right] \tag{16}$$

and the quantity

$$\widehat{Y}^* = N^* k + \frac{1}{\Delta^*}\widehat{\mathcal{E}}. \tag{17}$$

One can show that $\widehat{Y}^*$ is solution of $\partial_\nu \widehat{Y}^* + \left[A_\nu, \widehat{Y}^*\right] = 0$ with A the Lax connection corresponding to the dual fields $(\rho, \widehat{\sigma}, \Delta, N)$ in the triangular gauge, and therefore $\widehat{Y}^* = \Psi_{tr}\widehat{\mathcal{E}}\Psi_{tr}^{-1}$ where Ψ_{tr} is the wavefunction associated with A correctly normalized and in the triangular gauge. Now, if we consider the vacuum function dressed by some element g with a factorization such that the resulting wavefunction is in the triangular gauge, we can easily write the dual Ernst potential of the generated metric using the highest-weight vector defined in section 3.1,

$$\frac{1}{\Delta^*} \pm \mathrm{i}N^* = -\mathrm{i}\frac{\langle\Lambda_\pm|\Psi_V\left(g_-^{-1}\widehat{\mathcal{E}}g_+\right)\Psi_V^{-1}|\Lambda_\pm\rangle}{\langle\Lambda_\pm|\Psi_V g\Psi_V^{-1}|\Lambda_\pm\rangle}. \tag{18}$$

The conformal factor (and thus its dual value using (8)) can also be expressed using matrix elements,

$$\mathrm{e}^{2\widehat{\sigma}} = \langle\Lambda_+|\Psi_V g\Psi_V^{-1}|\Lambda_+\rangle\langle\Lambda_-|\Psi_V g\Psi_V^{-1}|\Lambda_-\rangle. \tag{19}$$

At this point, we still have two problems. On one hand, we have to fix the gauge if we want (18) to be valid, which requires choosing the factorization of g. On the other hand, we have obtained the dual fields not the fields themselves. We shall see that these two difficulties can be solved simultaneously.

4.2. *Vertex operators*

To solve these problems and to explicitly evaluate the metrics, we need a representation of $\mathcal{H}_{taf}$. For this purpose, we will use a vertex operator constructed using a $\mathbb{Z}_2$ twisted free bosonic field. Let us denote by $\widehat{X}(w)$ the bosonic field

$$\widehat{X}(w) = -\mathrm{i}\sum_{n\,\mathrm{odd}} p_{-n}\frac{w^n}{n} \qquad \text{with} \quad [p_n, p_m] = n\delta_{n+m,0}.$$

The operators p_n generate a Fock space, we denote by $|0\rangle$ its vacuum: $p_n|0\rangle = 0$ for $n > 0$. For any charge u, let $W_u(w)$ be the vertex operators:

$$W_u(w) = :\exp(-\mathrm{i}u\widehat{X}(w)):. \tag{20}$$

The colons refer to the normal ordering which amounts to moving to the right the oscillators p_n with n positive. The N-point functions of these vertex operators are given by

$$\left\langle\prod_p W_{u_p}(w_p)\right\rangle = \prod_{p,q}\left(\frac{w_p - w_q}{w_p + w_q}\right)^{u_p u_q/2}. \tag{21}$$

The representation of the affine Lie algebra on the Fock space is specified by the relation

$$\mathrm{i}w\frac{\mathrm{d}\widehat{X}(w)}{\mathrm{d}w} = \sum_{n\ \mathrm{odd}}(\sigma^z\lambda^n)w^{-n}$$
$$\pm\mathrm{i}W_2(w) = 2\sum_{n\ \mathrm{even}}((\sigma^+ - \sigma^-)\lambda^n)w^{-n} - 2\sum_{n\ \mathrm{odd}}((\sigma^+ + \sigma^-)\lambda^n)w^{-n}. \tag{22}$$

The highest-weight vectors $|\Lambda_\pm\rangle$ are identified with the vacuum vector $|0\rangle$ (the two representations are specified by the sign in front of $W_2(w)$).

Any element g of H_{taf} can be represented as a product of the vertex operator

$$g \equiv \prod_{p=1}^{m} W_{u_p}(w_p)\cdot\prod_{j=1}^{n}\left(1+\mathrm{i}y_j W_2(w_j)\right) \tag{23}$$

and the quantity $\widehat{\mathcal{E}}$ is equivalent to $\pm\mathrm{i}W_2(1)$. Thus, using (18) and (19), we associate with each metric two sets of parameters $\{(u_p, w_p)\}$ and $\{(y_j, w_j)\}$.

5. Explicit metric expressions

This section is devoted to the resolution of the previously exposed difficulties and to the final expressions of the metric elements [12]. Here we will only use the vertex operator representation. Most of the expressions are given up to constant factors that do not matter when evaluating the Ernst potential.

5.1. Solving the factorization problem

The factorization problem is closely related to the duality one. It can be shown that in the triangular gauge, we shall have

$$\varphi_\pm = \tfrac{1}{2}\left(\varphi \pm \varphi^*\right) \tag{24}$$

where φ and φ^* are the $SO(2)$ parameters of the metric and its dual as defined in section (4.1). The duality conditions (8) can be expressed as a system of equations similar to Hirota's equations, involving matrix elements of g and g^*. When solving these equations, we obtain the following formulae:

$$g = \text{constant}\ \prod_{p=1}^{m} W_{u_p}(w_p)\cdot\prod_{j=1}^{n}\left(1+\mathrm{i}y_j W_2(w_j)\right) \tag{25}$$

$$g^* = \text{constant}\ W_{-1}(1)\cdot\prod_{p=1}^{m} W_{-u_p}(w_p)\cdot\prod_{j=1}^{n}\left(1+\mathrm{i}y_j W_{-2}(w_j)\right). \tag{26}$$

Thus, duality is equivalent (up to a $W_{-1}(1)$ factor) to a change of the sign of the vertex operator charges. The factorization problem is solved by the two following identities:

$$g_-^{-1}W_2(1)g_+ = W_2(1)\cdot\prod_{p=1}^{m} W_{u_p}(w_p)\cdot\prod_{j=1}^{n}\left(1+\mathrm{i}y_j W_2(w_j)\right) \tag{27}$$

$$g_-^{*-1}W_2(1)g_+^* = W_1(1)\cdot\prod_{p=1}^{m} W_{-u_p}(w_p)\cdot\prod_{j=1}^{n}\left(1+\mathrm{i}y_j W_{-2}(w_j)\right). \tag{28}$$

The proof is based on equation (24), using a recurrence on the number of involved vertex operators.

5.2. Determinant formulae

Now we have all the tools needed to express metrics as a product of the determinant. In this section, we will choose coordinates such that ρ is equal to t to have simpler expressions. Using the notation

$$\langle g\rangle = \langle\Lambda_+|\Psi_V g\Psi_V^{-1}|\Lambda_+\rangle \qquad \text{and} \qquad \langle Wg\rangle = \langle\Lambda_+|\Psi_V g_-^{-1}W_2(1)g_+\Psi_V^{-1}|\Lambda_+\rangle \tag{29}$$

the metric elements are equivalent to

$$\mathrm{e}^{2\widehat{\sigma}} = |\langle g\rangle|^2 \qquad \mathrm{e}^{2\widehat{\sigma}^*} = |\langle g^*\rangle|^2 \tag{30}$$

$$\mathrm{e}^{2\widehat{\sigma}}G_{22} = \sqrt{\rho}\,|\langle g^*\rangle|^2 \qquad \mathrm{e}^{2\widehat{\sigma}^*}G_{22}^* = \sqrt{\rho}\,|\langle g\rangle|^2 \tag{31}$$

$$\mathrm{e}^{2\widehat{\sigma}}G_{12} = \sqrt{\rho}\,\mathrm{Im}\left[\langle g^*\rangle\,\overline{\langle Wg^*\rangle}\right] \qquad \mathrm{e}^{2\widehat{\sigma}^*}G_{12}^* = \sqrt{\rho}\,\mathrm{Im}\left[\langle g\rangle\,\overline{\langle Wg\rangle}\right] \tag{32}$$

$$\mathrm{e}^{2\widehat{\sigma}}G_{11} = \sqrt{\rho}\,|\langle Wg^*\rangle|^2 \qquad \mathrm{e}^{2\widehat{\sigma}^*}G_{11}^* = \sqrt{\rho}\,|\langle Wg\rangle|^2. \tag{33}$$

The various vacuum expectation values can be evaluated using the N-point function (21),

$$\langle g\rangle = \Omega\cdot\tau(Y_j|\mu_j) \qquad \langle g^*\rangle = \rho^{1/4}\,\Omega^*\cdot\tau(Y_j\,B_j|\mu_j) \tag{34}$$

$$\langle Wg\rangle = \rho\,\Omega_w\cdot\tau(Y_jB_j^2|\mu_j) \qquad \langle Wg^*\rangle = \rho^{1/4}\,\Omega_w^*\cdot\tau(Y_j\,B_j^{-1}|\mu_j). \tag{35}$$

The τ-function is defined by

$$\tau(Y|\mu) = \det{}_{n\times n}[1+\mathrm{i}V] \qquad \text{with} \quad V_{ij} = \frac{2\mu_i Y_j}{\mu_i+\mu_j} \tag{36}$$

with the following parameters

$$X_j = \rho\,\frac{w_j^2-1}{(w_j-z)^2+\rho^2} \qquad \text{and} \qquad \mu_j^2 = \frac{(w_j-z)+\rho}{(w_j-z)-\rho} \tag{37}$$

$$Y_j = y_j\,X_j\left(\prod_{p=1}^{m}B_{jp}^{u_p}\right) \qquad \text{with} \quad B_{jp} = \frac{\mu_j-\mu_p}{\mu_j+\mu_p} \quad \text{and} \quad B_j = \frac{1-\mu_j}{1+\mu_j}. \tag{38}$$

Finally, the prefactors Ω and Ω_w are given by

$$\begin{aligned}
\Omega &= \left(\prod_{p=1}^{m}X_p^{u_p^2/4}\right)\left(\prod_{p<q}B_{pq}^{u_pu_q/2}\right) \qquad & \Omega^* &= \left(\prod_{p=1}^{m}B_p^{u_p/2}\right)\Omega \\
\Omega_w &= \left(\prod_{p=1}^{m}B_p^{u_p}\right)\Omega \qquad & \Omega_w^* &= \left(\prod_{p=1}^{m}B_p^{-u_p/2}\right)\Omega.
\end{aligned} \tag{39}$$

As we mentioned in the introduction, the metric formulae have no quadrature left. They are expressed in quite a compact way using determinants and provide the metric and its dual at the same time. As an example of an application, if we take the following sets of parameters:

$$\{(z_p, u_p)\} = \{(1,-1), (+\infty,-1), (-1,-1)\}$$

and

$$\{(z_j, y_j)\} = \left\{\left(1, -\frac{q}{4(1+p)}\right), \left(-1, \frac{q}{4(1+p)}\right)\right\}$$

we obtain the Chandrasekhar–Xantopoulos solution [14] describing collisions of two impulsive gravitational waves with non-collinear polarization vectors.

6. Poisson algebra and r-matrix formulation

We have seen that the Lax connection is the key to quite simple methods for generating the solution. One could ask whether it could also be the starting point of a new method for quantizing the model. In this section, we will focus on the classical structure and the surprisingly simple form of the modified Yang–Baxter equations.

6.1. Canonical brackets

First of all, let us describe our phase space. As in [8], it is defined by the canonical variables $P_x, Q_x, \rho, \widehat{\sigma}$ and their associated momenta $\Pi_P, \Pi_Q, \Pi_\rho, \Pi_{\widehat{\sigma}}$. Here we use the canonical Poisson brackets to define the symplectic structure ($\left\{\rho(x), \Pi_\rho(y)\right\} = \delta(x-y)$ and so on).

To make contact with the model, we have to express the quantities P_t and Q_t in terms of the canonical variables. We define the variable P_t as

$$-\rho P_t = \partial_x \Pi_P + [Q_x, \Pi_P] + \left[P_x, \Pi_Q\right]. \tag{40}$$

Q_t will be considered to have vanishing brackets with all variables. We also need the generator of the local $SO(2)$ gauge transformation

$$-\Phi = \partial_x Q_x + \left[Q_x, \Pi_Q\right] + [P_x, \Pi_P] \equiv 0. \tag{41}$$

6.2. Modified Yang–Baxter equations

We will use the standard index-free tensor notation. For some element X, we define $X_1 \equiv X \otimes I$ and $X_2 \equiv I \otimes X$. We will also introduce the decomposition of the Casimir element $\mathcal{C}_{12}$ of $sl(2, R)$: $\mathcal{C}_{12} = c_{12} + d_{12}$, with $c_{12} = T^\alpha \otimes T_\alpha$ and $d_{12} = T^a \otimes T_a$. The validity of this decomposition is due to the orthogonality of generators with respect to the Killing form.

The Poisson brackets of the Lax connection are those of a non-ultralocal theory [15] with an $SO(2)$ gauge invariance. They are written as

$$\begin{aligned}\{A_{x1}(x), A_{x2}(y)\} = {} & \frac{1}{\rho(x)}\,\delta(x-y)\left(\left[r^\epsilon_{12}, A_{x1}(x)\right] + \left[s^\epsilon_{12}, A_{x2}(x)\right]\right) \\ & + \left(\frac{1}{\rho(x)}\,s_{12} - \frac{1}{\rho(y)}\,r_{12}\right)\partial_x\delta(x-y) \\ & - \frac{1}{8}\frac{1}{\rho^2(x)}\,\delta(x-y)\ [U_{12}, \Phi_1(x) - \Phi_2(x)]\end{aligned} \tag{42}$$

where

$$r^\pm_{12} = \frac{1}{2}\frac{(1-\lambda_1^2)(1-\lambda_2^2)}{\lambda_1^2-\lambda_2^2}c_{12} + \frac{1}{2}\frac{\lambda_1\lambda_2^{-1}(1-\lambda_2^2)^2}{\lambda_1^2-\lambda_2^2}d_{12} \mp \tfrac{1}{2}\left(E_\pm \otimes k + \tfrac{1}{2}k \otimes (E_+ + E_-)\right) \tag{43}$$

$$s^\pm_{12} = \frac{1}{2}\frac{(1-\lambda_1^2)(1-\lambda_2^2)}{\lambda_1^2-\lambda_2^2}c_{12} + \frac{1}{2}\frac{\lambda_1^{-1}\lambda_2(1-\lambda_1^2)^2}{\lambda_1^2-\lambda_2^2}d_{12} \mp \tfrac{1}{2}\left(k \otimes E_\mp + \tfrac{1}{2}(E_+ + E_-) \otimes k\right) \tag{44}$$

$$U_{12} = d_{12}(\lambda_1 - \lambda_1^{-1})(\lambda_2 - \lambda_2^{-1}). \tag{45}$$

The rational functions that appear in (43) and (44), only have a meaning as formal power series. So, whether we choose $|\lambda_1| < |\lambda_2|$ or $|\lambda_1| > |\lambda_2|$ when developing, we obtain two different sets of matrices (here a + convention refers to the case $|\lambda_1| < |\lambda_2|$). The validity of the brackets (42) (antisymmetry, independence with respect to the convention and Jacobi identity) is based on some relations verified by the constant involved matrices. The only non-pure c-number relation is

$$[r^\epsilon_{12} - r^{-\epsilon}_{12}, A_{x1} + A_{x2}] = -\left(\rho^{-1}\partial_x\rho\right)(r^\epsilon_{12} - r^{-\epsilon}_{12}). \tag{46}$$

Note that in a more traditional case (such as Toda's field theory), the difference between the r-matrices taken in two different conventions is proportional to the Casimir operator which is obviously not the case here. The other relations are

$$\begin{aligned} r_{12}^{\epsilon} &= -s_{21}^{-\epsilon} \\ r_{12}^{\epsilon} - r_{12}^{-\epsilon} &= s_{12}^{\epsilon} - s_{12}^{-\epsilon} \\ U_{12} &= U_{21}. \end{aligned} \tag{47}$$

In particular, we have two identities that allow one to verify the Jacobi identity. These modified Yang–Baxter equations can be written as

$$[r_{12}^{\epsilon_1}, r_{23}^{\epsilon_2}] + [s_{23}^{\epsilon_2}, s_{31}^{\epsilon_3}] + [s_{31}^{\epsilon_3}, r_{12}^{\epsilon_1}] - \tfrac{1}{2}k_2 s_{31}^{\epsilon_3} - \tfrac{1}{2}k_3 r_{12}^{\epsilon_1} - \tfrac{1}{4}[U_{23}, c_{12}] = 0 \tag{48}$$

$$\left[r_{23}^{\epsilon_2}, U_{12}\right] + [s_{23}^{\epsilon_2}, U_{13}] + \tfrac{1}{2}k_3 U_{12} - \tfrac{1}{2}k_2 U_{13} = 0. \tag{49}$$

The choice of the three conventions has to be consistent. This condition can be written as $|\epsilon_1 + \epsilon_2 + \epsilon_3| = 1$. What is quite remarkable is that the factorization of the coordinate dependence (through ρ) in (42) implies pure c-number equations for the modified Yang–Baxter equations. Introducing a Virasoro algebra allowed us to eliminate the coordinate dependence of these equations. Equations (48) and (49) can be interpreted as consistency conditions for linear Poisson brackets (see [13]), but we still lack a quadratic interpretation. Note that these calculations can easily be generalized to $\boldsymbol{G}/\boldsymbol{H}$ coset space σ-models coupled to two-dimensional gravity and a dilaton.

A different approach to this problem based on another Lax formulation with moving poles has been described in [8].

6.3. Classical observables

By definition, classical observables are functionals of the phase space variables that commute with the constraints. In our model, there are three constraints: two constraints associated with invariance under diffeomorphisms in the t and z directions whose generators are, respectively,

$$\mathcal{H} = -\Pi_\rho \Pi_{\widehat{\sigma}} - \partial_x \rho \partial_x \widehat{\sigma} + \tfrac{1}{2}\rho \operatorname{tr}\left(P_t^2 + P_x^2\right) + \operatorname{tr}(Q_t \Phi) \tag{50}$$

and

$$\mathcal{P} = \Pi_\rho \partial_x \rho + \Pi_{\widehat{\sigma}} \partial_x \widehat{\sigma} + \rho \operatorname{tr}(P_t P_x) + \operatorname{tr}(Q_x \Phi) \tag{51}$$

and one associated with invariance under local $SO(2)$ gauge transformations generated by Φ (see formula (41)). Thus, in our model, an observable $\mathcal{O}$ has to satisfy

$$\{\mathcal{H}(x), \mathcal{O}(y)\} = 0 \qquad \{\mathcal{P}(x), \mathcal{O}(y)\} = 0 \qquad \text{and} \qquad \{\Phi(x), \mathcal{O}(y)\} = 0.$$

The idea is to consider the monodromy matrix $\Psi(x, y) = \Psi(x)\Psi^{-1}(y)$ between two particular points, guided by the picture of cylindrical symmetry. First, we suppose there is a point x_∞ where ρ tends to ∞ and the metric is equivalent to flat Minkowski space. This is equivalent to saying that the metric is asymptotically flat. The other natural and usual point we shall consider is when ρ tends to 0 (as x goes to x_0) and behaves like the usual radial coordinates. With these boundary conditions, it can be proved that $\Psi_V^{-1}(x_0)\Psi(x_0, x_\infty)$ commutes with the diffeomorphism constraints. Deducing objects that are also invariant under $SO(2)$ gauge transformations is quite obvious. We only need to conjugate the previous quantity with ζ defined by $\partial_\mu \zeta + Q_\mu \zeta = 0$. Thus, the object

$$\widetilde{\Psi}(x_0, x_\infty) = \zeta^{-1}(x_0)\Psi_V^{-1}(x_0)\Psi(x_0, x_\infty)\zeta(x_\infty) \tag{52}$$

is an observable and since it is an operator, it provides an infinite set of physical observables.

To go further, we shall need to decipher the Poisson algebra of these observables. Unfortunately, the Poisson brackets for two monodromy matrices between two identical points are ill-defined. Such a difficulty also appears in other non-ultralocal theories (see [16, 17]). However, we hope that with some reasonable boundary conditions we will manage to find a way to solve it, as in [8].

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